On integrable potential perturbations of the Jacobi problem for the geodesics on the ellipsoid

1996 J. Phys. A: Math. Gen. 29 L317

(http://iopscience.iop.org/0305-4470/29/13/002)

J. Phys. A: Math. Gen. **29** (1996) L317–L321. Printed in the UK

LETTER TO THE EDITOR

On integrable potential perturbations of the Jacobi problem for the geodesics on the ellipsoid

Vladimir Dragovich†
Mathematical Institute SANU, Kneza Mihaila 35, 11000 Belgrade, Yugoslavia

Received 27 February 1996

Abstract. All integrable mechanical systems describing motion of a particle on an ellipsoid surface in three-dimensional space are described in the class of the Loran polynomial potentials. Two countable families of the basic solutions are obtained. Explicit formulae are given. The limit, when the considered system goes into the billiard system within an ellipse, is analysed, and the results are compared with those obtained previously, in relation to the billiard system.

1. Introduction

Completely integrable Hamiltonian systems are very rare but also very important. One of the most celebrated examples is the system describing a particle moving under inertia on an ellipsoid. In more geometrical language, it is the problem of the geodesics on the ellipsoid. As is well known, Jacobi proved the integrability of this system by introducing elliptical coordinates in which the equations of motion were separated.

Investigation of the integrability of perturbations of that system started with Jacobi himself, and since then few solutions have been found (see [1, 2] and the bibliography given therein).

In a recent paper on the subject [3], Kozlov proved the integrability of the n-dimensional problem with the addition of a potential force with the following potential:

$$V = \frac{k}{2}\sum_{s=0}^{n} x_s^2 + \sum_{r=0}^{n} \frac{\alpha_r}{x_r^2}. \tag{1}$$

In particular, he analysed the case of the ellipsoid in $\mathbb{R}^3$. Let us recall that in the limit, when the smallest axis goes to zero, the geodesics go into billiard trajectories within an ellipse. Kozlov investigated the limit of the above potentials and obtained the integrable billiards within an ellipse.

Using these ideas we have found a countable family of integrable potentials of the billiard system in [4]. In this paper we extend the family (1) of integrable potentials in the case of the ellipsoid in $\mathbb{R}^3$. We give explicit formulae for two countable families of the basic solutions. Then we compare the limit of these families with the solutions obtained in [4] for the billiard case.

† E-mail: vladad@mi.sanu.ac.yu

0305-4470/96/130317+05$19.50

2. The basic equations of Kozlov's method

Let us recall that a n-dimensional Hamiltonian system is completely integrable if it has n functionally independent integrals of motion that are in involution. Since the systems we consider here are two-dimensional, it is enough to prove the existence of only one integral, independent of the Hamiltonian (see [5]).

The system of a particle moving under inertia on the ellipsoid

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1 \tag{2}$$

has an additional integral of motion, found by Joahimstal [3],

$$I = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)\left(\frac{\dot{x}^2}{a} + \frac{\dot{y}^2}{b} + \frac{\dot{z}^2}{c}\right).$$

If the system is moving under the influence of a force with potential V the equations of motion are

$$\ddot{x} = \lambda x/a - V_x \qquad \ddot{y} = \lambda y/b - V_y \qquad \ddot{z} = \lambda z/c - V_z \tag{3}$$

where λ can be determined from

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)\lambda = \frac{x}{a}V_x + \frac{y}{b}V_y + \frac{z}{c}V_z - \frac{\dot{x}^2}{a} - \frac{\dot{y}^2}{b} - \frac{\dot{z}^2}{c}.$$

Kozlov's idea was to analyse whether equations (3) allow an integral F of the form

$$F = I + f$$

where f is a function depending only on the coordinates. From the condition $\dot{F} = 0$, which is equivalent to

$$2\left(\frac{x}{a}V_x + \frac{y}{b}V_y + \frac{z}{c}V_z\right)\left(\frac{x\dot{x}}{a^2} + \frac{y\dot{y}}{b^2} + \frac{z\dot{z}}{c^2}\right) - 2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) \times \left(\frac{\dot{x}}{a}V_x + \frac{\dot{y}}{b}V_y + \frac{\dot{z}}{c}V_z\right) + f_x\dot{x} + f_y\dot{y} + f_z\dot{z} = 0$$

he derived the system of three partial differential equations

$$2\left(\frac{x}{a}V_x + \frac{y}{b}V_y + \frac{z}{c}V_z\right)\frac{x}{a^2} - 2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)\frac{V_x}{a} = -f_x$$

$$2\left(\frac{x}{a}V_x + \frac{y}{b}V_y + \frac{z}{c}V_z\right)\frac{y}{b^2} - 2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)\frac{V_y}{b} = -f_y$$

$$2\left(\frac{x}{a}V_x + \frac{y}{b}V_y + \frac{z}{c}V_z\right)\frac{z}{c^2} - 2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)\frac{V_z}{c} = -f_z. \tag{4}$$

We are going to study the solutions of the corresponding system of compatibility conditions for equations (4),

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)V_{xy}\frac{a-b}{ab} - 3\frac{y}{b^2}\frac{V_x}{a} + 3\frac{x}{a^2}\frac{V_y}{b} + \left(\frac{x^2}{a^3} - \frac{y^2}{b^3}\right)V_{xy} + \frac{xy}{ab}\left(\frac{V_{yy}}{a} - \frac{V_{xx}}{b}\right) + \frac{zx}{ca^2}V_{zy} - \frac{zy}{cb^2}V_{zx} = 0$$

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)V_{yz}\frac{b-c}{bc}-3\frac{z}{c^2}\frac{V_y}{b}+3\frac{y}{b^2}\frac{V_z}{c}+\left(\frac{y^2}{b^3}-\frac{z^2}{c^3}\right)V_{yz}$$
$$+\frac{yz}{bc}\left(\frac{V_{zz}}{b}-\frac{V_{yy}}{c}\right)+\frac{xy}{ab^2}V_{xz}-\frac{xz}{ac^2}V_{xy}=0$$
$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)V_{zx}\frac{c-a}{ac}-3\frac{x}{a^2}\frac{V_z}{c}+3\frac{z}{c^2}\frac{V_x}{a}+\left(\frac{z^2}{c^3}-\frac{x^2}{a^3}\right)V_{zx}$$
$$+\frac{xz}{ac}\left(\frac{V_{xx}}{c}-\frac{V_{zz}}{a}\right)+\frac{zy}{c^2b}V_{xy}-\frac{yx}{ba^2}V_{yz}=0 \qquad (5)$$

in the form of the Loran polynomials:

$$V(x,y,z)=\sum_{r\geqslant n,m,p\geqslant s}a_{m,n,p}(\lambda)x^ny^mz^p \qquad s,r\in\mathbb{R}. \qquad (6)$$

By putting (6) into (5) we get the system of difference equations:

$$a_{m-2,n,p}\left(\frac{n}{a^2}\left(\frac{m+n}{b}+\frac{p}{c}\right)\right)-a_{m,n-2,p}\left(\frac{m}{b^2}\left(\frac{m+n}{a}+\frac{p}{c}\right)\right)$$
$$=mna_{m,n,p-2}\frac{b-a}{c^2ab}$$
$$a_{m,n-2,p}\left(\frac{p}{b^2}\left(\frac{n+p}{c}+\frac{m}{a}\right)\right)-a_{m,n,p-2}\left(\frac{n}{c^2}\left(\frac{n+p}{b}+\frac{m}{a}\right)\right)$$
$$=npa_{m-2,n,p}\frac{c-b}{a^2cb}$$
$$a_{m-2,n,p}\left(\frac{p}{a^2}\left(\frac{m+p}{c}+\frac{n}{b}\right)\right)-a_{m,n,p-2}\left(\frac{m}{c^2}\left(\frac{m+p}{a}+\frac{n}{b}\right)\right)$$
$$=mpa_{m,n-2,p}\frac{c-a}{b^2ca}. \qquad (7)$$

Lemma 1. System (7) with variables $a_{m-2,n,p}$, $a_{m,n-2,p}$ and $a_{m,n,p-2}$ is singular for arbitrary m, n, p, a, b, c.

Let us call the *level* of the element $a_{m,n,p}$ the sum $m+n+p$, and the *degree* of the element $a_{m,n,p}$ of the fixed level the sum $m+n$.

Lemma 2. Among the nonzero elements of the fixed level of the minimal degree has $a_{m,n,p}$ with $m=0$ or $n=0$.

Lemma 3. If m_0, p are such that $m_0\neq 0$ and $a_{m_0,0,p}$ is the nonzero element of the minimal degree of the level $k=m_0+p$ for general a, b, c then $k=-2$.

From now on, we will assume $a\neq b\neq c$.

3. The case $\mathbf{m_0\neq 0}$

If $m_0\neq 0$ then according to lemma 3 the nonzero elements are of level -2.

Lemma 4. (a)

$$a_{m_0,2k,-m_0-2-2k}=(-1)^k\frac{c^k(c-a)^k}{b^k(b-a)^k}\frac{\prod_{i=1}^k(m_0+2i)}{\prod_{i=1}^k 2i}a_{m_0,0,-m_0-2}$$

(b)

$$m_0=-2k, k\in N.$$

We shall call the potential *one parametric* if only one of the elements $a_{m,0,p}$ is different from zero.

Example 1. The simplest new integrable potential has the formula

$$\tilde{W}_2(x, y, z, \alpha) = \alpha x^{-4} z^2 + \frac{c(c-a)}{b(b-a)} \alpha x^{-4} y^2.$$

In order to describe one-parametric solutions we introduce the following function:

$$K(m_0, 2n) = 1 \qquad K(m_0 + 2, 2n) = n \qquad n \geqslant 2$$

and by induction

$$K(m_0 + 2k, 2n) = \sum_{s=k-1}^{n} K(m_0 + 2(k-1), 2s).$$

Some elementary combinatorics gives us

Lemma 5. The function K defined above has the following expression:

$$K(m_0 + 2k, 2n) = \binom{n+k-1}{k}.$$

Now, we have

Theorem 1. The general formula for the elements of one parametric solutions with $a_{m_0,0,-m_0-2} = \alpha$ is

$$a_{m_0+2k,2s,-m_0-2-2(k+s)} = (-1)^s \binom{s+k-1}{k} \times \frac{c^{k+s}(c-a)^s(c-b)^k \prod_{i=1}^{k+s}(m_0+2i)}{b^k a^s (b-a)^{k+s} \prod_{i=1}^{s} 2i \prod_{j=1}^{k}(m_0+2j)}$$

where $k \leqslant s$, and $m_0 + 2(k+s) < 0$.

Example 2. Using the above formula we give a more complicated integrable potential:

$$\tilde{W}_3(x, y, z, \alpha) = \alpha x^{-6} z^4 + 2\alpha \frac{c(c-a)}{b(b-a)} x^{-6} y^2 z^2 + \alpha \frac{c^2(c-a)^2}{b^2(b-a)^2} x^{-6} y^4 + \alpha \frac{c^2(c-a)(c-b)}{ab(b-a)^2} x^{-4} y^2.$$

4. The case $\mathbf{m_0 = 0}$

If the nonzero element of the smallest degree is of the form $a_{0,0,p}$ it is easy to find the general formula for the nonzero elements. It is given in the next theorem.

Theorem 2. If $a_{0,0,p}$ is the nonzero element of the smallest degree then $p = 2k$, $k \in N$, and

$$a_{2i,2j,2k-2(i+j)} = \frac{k!}{i!\,j!\,(k-i-j)!} (c/a)^i (c/b)^j a_{0,0,2k} \qquad i + j \leqslant k.$$

Example 3.

$$\tilde{W}_2' = \alpha \frac{c^2}{a^2} x^4 + \alpha \frac{c^2}{b^2} y^4 + \alpha z^4 + 2\alpha \frac{c}{a} x^2 z^2 + 2\alpha \frac{c^2}{ab} x^2 y^2 + 2\alpha \frac{c}{b} y^2 z^2.$$

Finally, we get

Theorem 3. Solutions of the form (6) of the system (5) are the linear combinations of the solutions described in theorems 1 and 2.

5. The billiard potentials as a limit of the Jacobi potentials

It is well known [6] that in the limit when the smallest axis c goes to zero, the considered system goes into the billiard system within an ellipse. In a previous work ([4]) we described potential perturbations of this system, in the class of the Loran polynomials. (Some other integrable perturbations of the system were given in [7, 8].) Now we want to illustrate the correspondence between solutions obtained here and those from ([4]).

Let us express z in terms of x, y from the ellipsoid equation (2),

$$z^2 = c\left(1 - \frac{x^2}{a} - \frac{y^2}{b}\right)$$

and insert it in the potential given in example 1:

$$\tilde{W}_2(x, y, c) = \alpha c x^{-4}\left(1 - \frac{x^2}{a} - \frac{y^2}{b}\right) + \alpha\frac{c(c-a)}{b(b-a)}x^{-4}y^2.$$

By computing the limit we obtain

$$\begin{aligned}\lim_{c\to 0}\tilde{W}_2(x, y, c)/c &= \alpha x^{-4} + \alpha\frac{x^{-4}y^2}{a-b} - \frac{\alpha}{a}x^{-2}\\ &= W_2(x, y, \alpha) + W_1\left(x, y, -\frac{\alpha}{a}\right)\end{aligned}$$

where W_2, W_1 are integrable potential perturbations of the billiard system within an ellipse, obtained in [4] and [3], respectively.

References

[1] Arnold V I, Kozlov V V and Neishtadt A I 1988 *Dynamical Systems III* (Berlin: Springer) p 291

[2] Perelomov A M 1990 *Integrable Systems of Classical Mechanics and Lie Algebras* (Moscow: Nauka) p 237

[3] Kozlov V V 1995 Some integrable generalizations of the Jacobi problem on geodesics on the ellipsoid *Prikl. Mat. Meh.* **59** 3–9

[4] Dragovich V I On integrable potentials of the billiard system within ellipse to appear

[5] Arnold V I 1978 *Mathematical Methods of Classical Mechanics (Graduate Texts in Mathematics 60)* (Berlin: Springer) p 462

[6] Birhof J 1941 *Dynamical Systems* (Moscow: Gostehizdat) p 320

[7] Ramani A, Kalliterakis A, Grammaticos B and Dorizzi B 1986 Integrable curvilinear billiards *Phys. Lett.* **115A** 2. 25–8

[8] Grammaticos B and Papageorgiou V 1988 Integrable bouncing-ball models *Phys. Rev.* A **37** 5000–1